

# Noether identities in gravity theories with nondynamical backgrounds and explicit spacetime symmetry breaking

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Gravitational effective field theories with nondynamical backgrounds explicitly break diffeomorphism and local Lorentz invariance. At the same time, to maintain observer independence the action describing these theories is required to be mathematically invariant under general coordinate transformations and changes of local Lorentz bases. These opposing effects of having broken spacetime symmetries but invariance under mathematical observer transformations can result in theoretical inconsistency unless certain conditions hold. The consistency constraints that must hold originate from Noether identities associated with the mathematical observer invariances in the action. These identities are examined in detail and are used to investigate gravity theories with nondynamical backgrounds, including when a Stückelberg approach is used. Specific examples include gravity theories with fixed scalar or tensor backgrounds, Einstein-Maxwell theory with a fixed external current, and massive gravity.

## I. INTRODUCTION

The idea that Lorentz symmetry might not hold at all energy scales has received much attention in recent years due to ideas originating from quantum gravity, string theory, and physics at the Planck scale [1, 2]. In addition, open questions related to dark energy and dark matter have led to modified theories of gravity being proposed, where in many cases Lorentz invariance no longer holds as an exact symmetry. Numerous experiments searching for violations of Lorentz symmetry in particle and gravitational interactions have been carried out. In these tests, the Standard-Model Extension (SME) is widely used as the phenomenological framework [3–5], allowing sensitivities to Lorentz violation to be expressed as bounds on the SME coefficients [6].

If local Lorentz invariance (LLI) is violated in gravitational or particle interactions, it is expected that the effects of this at low energy must be very small, involving for example suppression by the Planck scale. It is also expected on physical grounds that observer independence should hold even if violations of LLI occur [3, 4]. For these reasons, Lagrangian-based effective field theory provides a suitable approach in theoretical investigations, where Lorentz-violating terms are incorporated in an observer-independent action as couplings between fixed background fields and conventional gravitational and matter fields.

In gravitational theories where violation of LLI is due to spontaneous symmetry breaking at a more fundamental level, the background fields represent dynamical vacuum solutions, which spontaneously break both LLI and diffeomorphism invariance. In this case, when Nambu-Goldstone and massive excitations about the vacuum solution are included, the invariance of the action under both spacetime symmetries still holds [7]. On the other hand, when a *nondynamical* background appears directly in the Lagrangian, the symmetry breaking is explicit. In this case, there are no dynamical excitations of the background field, and the action is not invariant under either

local Lorentz transformations or diffeomorphisms.

When spacetime symmetry breaking is explicit due to the presence of a nondynamical background, it is known that potential conflicts can occur between geometrical identities, conservation laws, and the dynamical equations of motion [4, 8]. These conflicts can lead to theoretical inconsistency unless certain conditions hold. In contrast, when the symmetry breaking is spontaneous, the potential conflicts are evaded since the background fields are dynamical. For these reasons most investigations using the SME assume spontaneous breaking of diffeomorphisms and LLI in the gravity sector. Nonetheless, there are examples of widely studied gravitational theories with nondynamical backgrounds that explicitly break local diffeomorphism invariance and LLI. Examples include Chern-Simons gravity [9], theories with spacetime-varying couplings [10–13], and massive gravity [14].

In gravitational theories with explicit breaking and nondynamical backgrounds, the consistency conditions that are usually verified are the equations,  $D_\mu T^{\mu\nu} = 0$  and  $T^{\mu\nu} = T^{\nu\mu}$ , where  $T^{\mu\nu}$  is the energy-momentum tensor obtained using either a metric or vierbein formalism. The nondynamical background fields enter these equations as part of  $T^{\mu\nu}$ .

In General Relativity (GR), these consistency conditions are obtained by combining the Noether identities associated with diffeomorphism invariance and LLI with the dynamical equations of motion for the gravitational and matter fields. The usual interpretation of the Noether identities in GR is that they provide relations that hold off shell between the Euler-Lagrange expressions for the matter and gravitational fields [15–19]. In the case of diffeomorphism invariance, this leads to the result that the four equations  $D_\mu T^{\mu\nu} = 0$  automatically hold when the matter fields are on shell. Similarly, in GR in a vierbein treatment the Noether identities associated with LLI provide off-shell relations involving anti-symmetric combinations of the matter and gravitational fields and their Euler-Lagrange expressions. On shell, these result in the six equations  $T^{\mu\nu} = T^{\nu\mu}$ .

In contrast, in a theory with a nondynamical background, both diffeomorphism invariance and LLI are explicitly broken, and the usual Noether identities stemming from these symmetries no longer apply. Nonetheless, the same consistency conditions,  $D_\mu T^{\mu\nu} = 0$  and  $T^{\mu\nu} = T^{\nu\mu}$ , must hold on shell even when a nondynamical background is present. In this case, it is because these equations follow directly from Einstein's equations,  $G^{\mu\nu} = 8\pi G T^{\mu\nu}$ , where  $G^{\mu\nu}$  is the Einstein tensor, combined with the contracted Bianchi identity,  $D_\mu G^{\mu\nu} = 0$ , and the symmetry relation  $G^{\mu\nu} = G^{\nu\mu}$ .

At the same time, it is important to realize that a form of Noether identities can still be found when there is explicit spacetime symmetry breaking. In this case, the identities stem from the requirement of observer independence, which provides the freedom to pick any coordinate system and any local Lorentz basis. Because of observer independence, the action is required to be mathematically invariant under both general coordinate transformations and changes of local Lorentz bases. This leads to Noether identities that can be derived using these observer invariances of the action. In this case, the identities involve not only the Euler-Lagrange expressions for the dynamical fields, but also for the nondynamical background fields. However, the nondynamical backgrounds do not have to obey their Euler-Lagrange equations. Therefore, the usual interpretation of the Noether identities does not hold, and a different interpretation is required when nondynamical background fields are present.

The goal of this paper is to examine the Noether identities that arise from the requirement of observer independence in gravitational theories that contain a nondynamical background. In addition, it is shown that making a detailed examination of these Noether identities can provide insight into the potential theoretical inconsistencies that can occur in a theory with explicit spacetime symmetry breaking. For example, when the equations  $D_\mu T^{\mu\nu} = 0$  or  $T^{\mu\nu} = T^{\nu\mu}$  are found not to hold in a gravitational theory, which either makes the theory inconsistent or imposes restrictions on the physical fields and/or geometry, the underlying reasons are often not readily apparent. In this case, it is advantageous to look directly at the structure of the Noether identities associated with observer independence in order to understand how the inconsistency arises or is possibly evaded.

This paper is organized as follows. The next section begins with a brief review of spacetime symmetry transformations and the usual interpretation of the Noether identities in GR. Since diffeomorphism invariance is manifest in a metric description while LLI remains hidden, a vierbein formalism is used to examine LLI. However, torsion is assumed to vanish in this investigation and only Riemann spacetimes are considered. Section III discusses diffeomorphism and LLI breaking in theories with nondynamical background fields. The Noether identities that hold as a result of observer independence are presented, and their interpretation is discussed in com-

parison to GR and theories with spontaneous spacetime symmetry breaking. Section IV looks at applications of these Noether identities, including specific examples of gravity theories with nondynamical backgrounds. Lastly, Section V gives a summary and conclusions.

## II. SPACETIME SYMMETRIES IN GR

In a Lagrangian description of GR, diffeomorphism invariance and LLI act effectively as local gauge symmetries that leave the action describing the theory unchanged. As a result, both of these local spacetime symmetries lead to Noether identities.

First, consider diffeomorphism invariance using a metric description. A generic form of the action in GR consists of an Einstein-Hilbert term and a matter term,

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R + \mathcal{L}(g_{\mu\nu}, f^\psi) \right]. \quad (1)$$

Here, units with  $\hbar = c = 8\pi G = 1$  are used. The matter term  $\mathcal{L}$  depends on the metric tensor,  $g_{\mu\nu}$ , and dynamical matter fields generically represented as tensors  $f^\psi$ , where  $\psi$  collectively denotes the relevant tensor indices. For simplicity, it is assumed that  $\mathcal{L}$  depends on  $f^\psi$  and at most first covariant derivatives of  $f^\psi$ .

Under local diffeomorphisms, with vector parameters,  $\xi^\mu$ , tensor fields transform with changes given as Lie derivatives, while the spacetime coordinates remain unchanged. Specifically,  $f^\psi \rightarrow f^\psi + \mathcal{L}_\xi f^\psi$ , and

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} = g_{\mu\nu} + D_\mu \xi_\nu + D_\nu \xi_\mu. \quad (2)$$

In GR,  $S$  is invariant under these transformations.

To reveal the LLI, a vierbein description is used, where the metric is defined in terms of the vierbein as

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \quad (3)$$

Here, Latin indices denote components with respect to a local Lorentz basis in the tangent plane at each spacetime point, while Greek indices are reserved for spacetime components.

For simplicity in this investigation, a vierbein formalism restricted to Riemann spacetime (with no torsion) is used, and spinor fields are not included. Therefore, the resulting Einstein equations have the same structure as in GR, except that contributions to  $T^{\mu\nu}$  can involve local matter fields as well as the vierbein, which both transform under local Lorentz transformations. It is in this way that LLI becomes a relevant local symmetry in GR.

Using such a vierbein approach, the action in GR becomes

$$S = \int d^4x e \left[ \frac{1}{2} R + \mathcal{L}(e_\mu^a, f_y^\psi) \right], \quad (4)$$

where  $e$  is the determinant of the vierbein. The tensors denoted here as  $f_y^\psi$  can carry both spacetime and local

indices, where  $y$  collectively denotes the local Lorentz indices.

Local Lorentz transformations are made with respect to a given local basis. An infinitesimal transformation of  $f^\psi_y$  has the form

$$f^\psi_y \rightarrow f^\psi_y + \frac{1}{2}\epsilon^{ab}(X_{[ab]})^x_y f^\psi_x, \quad (5)$$

where  $\epsilon_{ab} = -\epsilon_{ba}$  are the six infinitesimal parameters for the local Lorentz group, and  $(X_{[ab]})^x_y$  gives an irreducible representation. As a specific example, the vierbein,  $e_\mu^a$ , transforms as a contravariant vector under infinitesimal local Lorentz transformations. Putting in the appropriate representation, the transformation takes the form

$$e_\mu^a \rightarrow e_\mu^a + \epsilon^a_b e_\mu^b. \quad (6)$$

At the same time,  $e_\mu^a$  transforms as a covariant space-time vector under diffeomorphisms, and  $f^\psi_y$  transforms as a spacetime tensor with indices denoted by  $\psi$ . The action in (4) is then invariant under both diffeomorphisms and local Lorentz transformations.

### A. Observer transformations

The spacetime coordinate system and local Lorentz basis do not change under diffeomorphism and local Lorentz gauge transformations. Only the physical gravitational and matter fields transform. At the same time, the action in GR is mathematically invariant under both general coordinate transformations and rotations of the Lorentz bases in local frames. These mathematical invariances of the action are referred to here, respectively, as general coordinate invariance (GCI) and observer LLI.

In GR, it can be shown that infinitesimal general coordinate transformations,  $x^\mu \rightarrow x'^\mu = x^\mu - \xi^\mu$ , combined with an expansion of the Lagrangian in  $\xi^\mu$  and relabeling of coordinates, result in transformations of the fields that have the same mathematical form as the diffeomorphism gauge transformations. Likewise, in a vierbein treatment, if inverse parameters,  $-\epsilon_{ab}$ , are used to rotate the local Lorentz basis, the resulting infinitesimal transformations of local tensor components are mathematically the same as the local Lorentz transformations.

In GR, and in the absence of symmetry breaking, these can be viewed as active and passive transformations, which are inversely related. The gauge symmetries in fixed coordinate and local frames are the active transformations, while GCI and observer LLI are passive transformations that move between different observer frames.

### B. Noether Identities in GR

Historically, Noether proved two theorems that are important in field theory [15]. The first involves global symmetry groups, with transformation parameters that

are constant. It states that a conserved current can be associated with invariance of the action under a global symmetry. The second theorem involves local symmetry groups, with parameters that are spacetime dependent. It results in identities that must hold when a theory is invariant under a local symmetry group.

In GR, diffeomorphisms and LLI are local symmetries, and therefore according to the second theorem there are associated Noether identities [16–19]. However, in GR, the same mathematical transformations in the action can be obtained using either the active forms due to diffeomorphism invariance and LLI or using the passive transformations associated with GCI and observer LLI [20]. Thus, the same Noether identities can be obtained using either type of transformation in the context of GR.

At the same time, it is important to keep in mind that there are differences between active and passive space-time transformations. For example, the active transformations are symmetry transformations, where invariance of the action only holds when the symmetries are unbroken. It is for this reason that nondynamical background fields are not permitted in GR. On the other hand, any realistic action is required to be invariant under the observer transformations so that observer independence is maintained. In this case, invariance is imposed on the action regardless of whether there are background fields or not. Nonetheless, since GR does not allow fixed background fields, the Noether identities resulting from GCI and observer LLI are mathematically the same in the context of GR as those due to diffeomorphism invariance and LLI in GR.

The Noether identity stemming from diffeomorphism invariance in GR (or equivalently GCI) is obtained by performing field variations in the action given in (1), where the variations in the metric and matter fields are given as Lie derivatives, with local parameters  $\xi^\mu$ , acting on these fields. The invariance of the action gives

$$\int d^4x \left[ \frac{1}{2} \frac{\delta(\sqrt{-g}R)}{\delta g_{\mu\nu}} \mathcal{L}_\xi g_{\mu\nu} + \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} \mathcal{L}_\xi g_{\mu\nu} + \sqrt{-g} \frac{\delta\mathcal{L}}{\delta f^\psi} \mathcal{L}_\xi f^\psi \right] = 0. \quad (7)$$

Putting in the expressions for the Lie derivatives, which depend on the type of the tensor  $f^\psi$ , and performing integrations by parts, gives a generic expression

$$\int d^4x \sqrt{-g} \xi_\nu [D_\mu (G^{\mu\nu} - T^{\mu\nu}) + \frac{\delta\mathcal{L}}{\delta f^\psi} \gamma^{\psi\nu} + D_\mu (\frac{\delta\mathcal{L}}{\delta f^\psi} \gamma^{\psi\mu\nu})] = 0. \quad (8)$$

Here, the spacetime indices represented by  $\psi$  are summed, and the coefficients  $\gamma^{\psi\nu}$  and  $\gamma^{\psi\mu\nu}$  represent general functions of the field variables, which depend on the form of  $f^\psi$ .

The Noether identity associated with diffeomorphism invariance follows from requiring that the integral in (8) must vanish for all parameters  $\xi_\nu$  that vanish on the

boundary surfaces. This results in four identities that must be obeyed by the metric and matter fields, which have the generic form

$$D_\mu(G^{\mu\nu} - T^{\mu\nu}) + \frac{\delta\mathcal{L}}{\delta f^\psi} \gamma^{\psi\nu} + D_\mu\left(\frac{\delta\mathcal{L}}{\delta f^\psi} \gamma^{\psi\mu}\right) = 0, \quad (9)$$

The first term in these identities contains the Euler-Lagrange expression for the metric,  $(G^{\mu\nu} - T^{\mu\nu})$ , while  $\frac{\delta\mathcal{L}}{\delta f^\psi}$  represents the Euler-Lagrange expression for the tensor  $f^\psi$ . For example, if  $f^\psi$  is a contravariant vector with index  $\psi$  replaced by  $\alpha$ , and  $\mathcal{L}$  depends on both  $f^\alpha$  and its first derivative, then

$$\frac{\delta\mathcal{L}}{\delta f^\alpha} \equiv -D_\mu \left( \frac{\partial\mathcal{L}}{\partial D_\mu f^\alpha} \right) + \frac{\partial\mathcal{L}}{\partial f^\alpha}. \quad (10)$$

Note that as an identity, (9) holds off shell, i.e., the Euler-Lagrange expressions need not vanish.

The usual interpretation of this Noether identity in GR is that four of the field components in the theory are not dynamically independent from the others. Thus, if all but four Euler-Lagrange equations are set to zero, then the remaining four equations of motion must hold automatically. In particular, by combining (9) with the contracted Bianchi identity,  $D_\mu G^{\mu\nu} = 0$ , it also follows that  $D_\mu T^{\mu\nu} = 0$  automatically holds in GR when the matter fields are on shell, obeying  $\frac{\delta\mathcal{L}}{\delta f^\psi} = 0$ .

In GR, the loss of four independent degrees of freedom is expected as a result of the local gauge invariance under diffeomorphisms, where four field components can be fixed or set to zero by imposing gauge-fixing conditions. Such gauge-fixing conditions, or equivalently coordinate-fixing conditions, are in fact needed in order to be able to carry out and solve the initial-value problem in GR. Only with these extra conditions can an unambiguous evolution be determined for the dynamical degrees of freedom, subject to their initial constraints.

To examine Noether's identity associated with LLI, a vierbein approach can be used. The action in the absence of spin and torsion then has the form given in (4), and fields with local tensor indices transform, as in (5) for example, under local Lorentz transformations. In this case, the corresponding Noether identity is obtained by performing variations having the form of local Lorentz transformations. The resulting identity consists of six equations,

$$(G^{\mu\nu} - T^{\mu\nu})(e_{\mu a} e_{\nu b} - e_{\mu b} e_{\nu a}) + \frac{1}{2} \frac{\delta\mathcal{L}}{\delta f^\psi_y} (X_{[ab]})^x_y f^\psi_x = 0, \quad (11)$$

which hold off shell. When the matter fields are on shell, obeying  $\frac{\delta\mathcal{L}}{\delta f^\psi_y} = 0$ , multiplying by inverse vierbeins and using the symmetry of the Einstein tensor shows that it reduces to the requirement that the energy-momentum tensor must be symmetric, obeying  $T^{\mu\nu} = T^{\nu\mu}$ .

### III. NONDYNAMICAL BACKGROUNDS

When a field theory includes a nondynamical background field, the usual notion of an active symmetry transformation no longer applies. This is because the background field is understood to be unchangeable, and it therefore cannot be transformed. In this context, it is useful and more common to distinguish what are known as particle and observer transformations [4]. These are the types of transformations that are relevant when there is spacetime symmetry breaking.

Particle transformations are the same as active transformations when they act on any fields other than the background fields. However, when a particle transformation acts on a background field the background remains fixed and does not transform. Particle transformations therefore correspond to the physical transformations that can be performed in a laboratory, where it is not possible for an experimenter to alter the background fields. Notice that particle transformations keep the coordinate system and local Lorentz basis unchanged. In contrast, observer transformations are passive changes of the coordinates and local bases. Under observer transformations, all tensor components transform, including those of the background fields.

Observer independence requires that the Lagrangian must be a scalar under observer spacetime transformations. Thus, when couplings to a fixed background appear in a theory, it is the particle symmetries that are broken either spontaneously or explicitly, while GCI and observer LLI remain mathematical invariances of the action.

If the background field is dynamical, forming as a result of spontaneous breaking of the particle symmetries, then it satisfies the equations of motion as a vacuum solution. In this case, the interpretation of Noether's identities is the same as in GR. This is true as well when excitations about the background occur, since these include Nambu-Goldstone and massive modes that restore the spontaneously broken particle symmetries.

However, when a background field is nondynamical, particle diffeomorphisms and LLI are broken explicitly, and the Euler-Lagrange expressions for the background fields need not vanish. Therefore, the usual interpretation of the Noether identities does not apply, and instead a modified interpretation is needed.

#### A. Explicit Symmetry Breaking

To investigate Noether's identities with explicit diffeomorphism breaking, consider a gravitational theory with a fixed nondynamical background, which is denoted generically as a tensor  $\bar{k}_{\lambda\mu\nu\dots}$ . In a metric description, the action is

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R + \mathcal{L}(g_{\mu\nu}, f^\psi, \bar{k}_{\lambda\mu\nu\dots}) \right], \quad (12)$$

where  $\mathcal{L}$  is an observer scalar that depends on the dynamical matter fields,  $f^\psi$ , and their first derivatives, as well as on the nondynamical background,  $\bar{k}_{\lambda\mu\nu\dots}$ . With this form, the action  $S$  explicitly breaks particle diffeomorphisms while maintaining observer GCI.

To consider local Lorentz transformations, the theory can be redefined using dynamical vierbeins,  $e_\mu^a$ , in place of the metric. At the same time, the background field can be written in terms of tensor components given with respect to both the spacetime frame and the local Lorentz frame, where in the latter case they are denoted as  $\bar{k}_{abc\dots}$ . While the dynamical vierbein and matter fields transform under particle diffeomorphisms and local Lorentz transformations, the nondynamical background tensor,  $\bar{k}_{\lambda\mu\nu\dots}$ , remains fixed. Since the coordinate system and local basis do not change either under these particle transformations, the components of the background defined with respect to the local Lorentz basis,  $\bar{k}_{abc\dots}$ , must remain fixed as well. This means that these different fixed components must be related by a nondynamical vierbein, denoted as  $\bar{e}_\mu^a$ , which is also fixed under particle diffeomorphisms and local Lorentz transformations. The defining relation for the background vierbein is

$$\bar{k}_{\lambda\mu\nu\dots} = \bar{e}_\lambda^a \bar{e}_\mu^b \bar{e}_\nu^c \dots \bar{k}_{abc\dots}, \quad (13)$$

where every quantity in this expression remains fixed under particle diffeomorphisms and local Lorentz transformations.

The action replacing (12) can then be written as

$$S = \int d^4x e \left[ \frac{1}{2} R + \mathcal{L}(e_\mu^a, \bar{e}_\mu^a, f^\psi_y, \bar{k}_{abc\dots}) \right], \quad (14)$$

where  $\mathcal{L}$  now depends on both the physical vierbein,  $e_\mu^a$ , and the background vierbein,  $\bar{e}_\mu^a$ , in addition to the dynamical matter fields and the nondynamical background. This form of the action explicitly breaks diffeomorphisms and local Lorentz transformations, but it is mathematically invariant under observer general coordinate transformations and changes of local Lorentz bases, since all of the field components in  $\mathcal{L}$  transform under the observer transformations.

## B. Noether Identities with Fixed Backgrounds

The Noether identities that hold when fixed backgrounds are present stem from the observer invariances in the action. The identities are obtained by performing mathematical field variations in the action having the form of the local observer transformations. As in GR, both a metric formalism and a vierbein formalism can be considered.

### 1. Metric Formalism

In a metric formalism, the action is given in (12), and the Noether identity associated with observer GCI

is found to have the general form

$$D_\mu(G^{\mu\nu} - T^{\mu\nu}) + \frac{\delta\mathcal{L}}{\delta f^\psi} \gamma^{\psi\nu} + D_\mu \left( \frac{\delta\mathcal{L}}{\delta f^\psi} \gamma^{\psi\mu\nu} \right) + \frac{\delta\mathcal{L}}{\delta \bar{k}_{\alpha\beta\gamma\dots}} \lambda_{\alpha\beta\gamma\dots}^\nu + D_\mu \left( \frac{\delta\mathcal{L}}{\delta \bar{k}_{\alpha\beta\gamma\dots}} \lambda_{\alpha\beta\gamma\dots}^{\mu\nu} \right) = 0. \quad (15)$$

Here, the coefficients  $\gamma^{\psi\nu}$ ,  $\gamma^{\psi\mu\nu}$ ,  $\lambda_{\alpha\beta\gamma\dots}^\nu$  and  $\lambda_{\alpha\beta\gamma\dots}^{\mu\nu}$  denote functions of the field components, where their specific forms depend on how  $\mathcal{L}$  is defined. Note that these are off-shell equations that hold for all values of the fields.

The potential conflict that arises when diffeomorphisms are explicitly broken becomes evident when the dynamical matter fields are put on shell. Setting  $\frac{\delta\mathcal{L}}{\delta f^\psi} = 0$  and using the contracted Bianchi identity,  $D_\mu G^{\mu\nu} = 0$ , reveals that  $D_\mu T^{\mu\nu}$  can only vanish when the remaining two terms in the Noether identity vanish. At the same time, consistency with Einstein's equations requires that  $D_\mu T^{\mu\nu} = 0$  must hold on shell. Therefore theoretical consistency requires that on shell the following must hold:

$$\frac{\delta\mathcal{L}}{\delta \bar{k}_{\alpha\beta\gamma\dots}} \lambda_{\alpha\beta\gamma\dots}^\nu + D_\mu \left( \frac{\delta\mathcal{L}}{\delta \bar{k}_{\alpha\beta\gamma\dots}} \lambda_{\alpha\beta\gamma\dots}^{\mu\nu} \right) = 0. \quad (16)$$

However, unlike the matter fields, the nondynamical background fields need not have vanishing Euler-Lagrange equations, which allows

$$\frac{\delta\mathcal{L}}{\delta \bar{k}_{\alpha\beta\gamma\dots}} \neq 0. \quad (17)$$

Thus, (16), which contains the fixed background, becomes a consistency condition that must be satisfied by the dynamical fields in the theory.

In general it is possible for the four equations in (16) to have solutions because the loss of diffeomorphism invariance means that there are four additional degrees of freedom in comparison to GR. These are the degrees of freedom that in a gauge-invariant theory such as GR would normally be gauged away. However, explicit breaking no longer allows this. In a metric formalism, it is natural to let the metric tensor have the four additional degrees of freedom. In that case, (16) can be viewed as four constraints that can be satisfied by the metric due to its having four additional degrees of freedom.

At the same time, inconsistency can arise when (16) is found not to hold. For example, there must be sufficient coupling with the metric so that at least four components appear in these equations. If a specific ansatz for the metric is assumed, as in a spatially homogeneous and isotropic universe, there might not be enough degrees of freedom in the metric to satisfy (16). This can lead to certain geometries being ruled out in the presence of incompatible backgrounds. In other cases, if there are not enough metric components but matter couplings occur, this can put additional conditions on the matter fields, which in turn can lead to consistency issues in the matter sector.

Note that even if consistency is maintained, the interpretation of Noether's identity when there is explicit

breaking is fundamentally different from the interpretation in GR. For example, in GR the metric affects the curvature, which remains largely distinguishable from matter dynamical effects even when both have backreactions on the other. However, with explicit breaking, the metric affects not only the curvature but it must also absorb the physical backreactions that the fixed background tensor is unable to have.

It is also possible to interpret Noether's identity when a background field is present as providing a set of coordinate-fixing conditions for the metric. The specific form of the conditions depends on  $\mathcal{L}$  and the values given for  $\bar{k}_{\lambda\mu\nu\dots}$  in a particular coordinate frame. Normally GCI allows freedom in the choice of coordinates. However, with explicit breaking, when the components of a fixed background tensor are given specified values,  $\bar{k}_{\lambda\mu\nu\dots}$ , this must be done with respect to a particular coordinate frame. It is then the conditions from the Noether identity that makes the chosen coordinate frame compatible with the dynamical equations of motion. This in turn allows covariant energy-momentum conservation to hold.

As an illustrative example, if explicit breaking is caused by a term with  $\mathcal{L} = g^{\mu\nu}\eta_{\mu\nu}$  in the action, where  $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  in the chosen coordinate frame, then the conditions in (16) can be shown to reduce to the requirement that  $g^{\mu\nu}\Gamma_{\mu\nu}^\lambda = 0$ . In GR, these are the conditions for harmonic coordinates, which are commonly chosen as gauge-fixing conditions. The difference here is that they emerge as a result of the Noether identity associated with GCI and the choice of  $\mathcal{L}$  and the background field  $\eta_{\mu\nu}$ .

Theories with explicit diffeomorphism breaking differ from GR in other important respects as well. For example, theories with nondynamical backgrounds have a different constraint structure, and therefore the number of physical degrees of freedom can differ [21]. As a result, the propagation of gravitational excitations is typically modified. Based on these effects, models can be investigated and in some cases ruled out if they permit unphysical degrees of freedom (such as ghost modes) or if they contradict experiments. While these are serious issues concerning the physical viability of theories with nondynamical backgrounds, they are not explored further here. Instead, the focus here remains on the Noether identities, which in many respects are more restrictive than phenomenological tests, since the Noether identities provide consistency conditions that must hold purely on theoretical grounds.

## 2. Vierbein Formalism

Using a vierbein formalism, with the action given in (14), Noether identities associated with both GCI and observer LLI can be found.

The identity stemming from GCI has a structure similar to (15) except that the matter and background vierbein also carry local indices. For the matter fields these

are labeled generically using  $y$  to denote the set of local indices. As a result, the coefficients in the terms involving the matter fields,  $f_y^\psi$ , carry local indices as well, and these are written as  $\gamma^{\psi\nu}_y$  and  $\gamma^{\psi\mu\nu}_y$ . The background vierbein,  $\bar{e}_\mu^a$ , and local tensor,  $\bar{k}_{abc\dots}$ , are respectively vectors and scalars under general coordinate transformations, and their contributions can be written explicitly. The result is

$$\begin{aligned} D_\mu(G^{\mu\nu} - T^{\mu\nu}) + \frac{\delta\mathcal{L}}{\delta f_y^\psi} \gamma^{\psi\nu}_y + D_\mu \left( \frac{\delta\mathcal{L}}{\delta f_y^\psi} \gamma^{\psi\mu\nu}_y \right) \\ - D_\mu \left( \frac{\delta\mathcal{L}}{\delta \bar{e}_\mu^a} g^{\nu\alpha} \bar{e}_\alpha^a \right) + \frac{\delta\mathcal{L}}{\delta \bar{e}_\mu^a} D^\nu \bar{e}_\mu^a \\ + \frac{\delta\mathcal{L}}{\delta \bar{k}_{abc\dots}} D^\nu \bar{k}_{abc\dots} = 0. \end{aligned} \quad (18)$$

When the matter fields,  $f_y^\psi$ , are put on shell, and the contracted Bianchi identity is used, this identity shows that the last three terms involving  $\bar{e}_\mu^a$  and  $\bar{k}_{abc\dots}$  must vanish in order for covariant energy-momentum conservation to hold. Since in this case, the Euler-Lagrange expressions for the backgrounds need not vanish,

$$\frac{\delta\mathcal{L}}{\delta \bar{e}_\mu^a} \neq 0, \quad \frac{\delta\mathcal{L}}{\delta \bar{k}_{abc\dots}} \neq 0, \quad (19)$$

there is again the possibility of conflicts and inconsistency. To avoid these issues it must be that the physical vierbein, which has four additional modes due to the lack of diffeomorphism invariance, must take values that satisfy these conditions. The question of whether this happens consistently or not then largely depends on the extent to which vierbein couplings are included in the terms appearing in (18).

The Noether identity stemming from observer LLI can be found as well. It is obtained by performing infinitesimal observer Lorentz transformations in the action, where a general representation of the local Lorentz group is used for  $f_x^\psi$ , while vector and tensor representations are used, respectively, for  $\bar{e}_\mu^a$  and  $\bar{k}_{abc\dots}$ . The result is

$$\begin{aligned} (G^{\mu\nu} - T^{\mu\nu})(e_{\mu a} e_{\nu b} - e_{\mu b} e_{\nu a}) \\ + \frac{1}{2} \frac{\delta\mathcal{L}}{\delta f_y^\psi} (X_{[ab]})^x_y f_x^\psi + \left( \frac{\delta\mathcal{L}}{\delta \bar{e}_\mu^a} \bar{e}_{\mu b} - \frac{\delta\mathcal{L}}{\delta \bar{e}_\mu^b} \bar{e}_{\mu a} \right) \\ + \frac{\delta\mathcal{L}}{\delta \bar{k}_{cde\dots}} [(\eta_{ac} \bar{k}_{bde\dots} - \eta_{bc} \bar{k}_{ade\dots}) \\ + (\eta_{ad} \bar{k}_{cbe\dots} - \eta_{bd} \bar{k}_{cae\dots}) \\ + (\eta_{ac} \bar{k}_{cdb\dots} - \eta_{bc} \bar{k}_{cda\dots}) + \dots] = 0. \end{aligned} \quad (20)$$

When the matter fields are put on shell, obeying  $\frac{\delta\mathcal{L}}{\delta f_y^\psi} = 0$ , and the symmetry of the Einstein tensor is used, it follows that  $T^{\mu\nu}$  is symmetric only if the additional terms in (20) either cancel or vanish. Since Einstein's equations (in the absence of spin and torsion)

give the result that  $T^{\mu\nu}$  must be symmetric, theoretical consistency requires that the additional terms associated with  $\bar{e}_\mu^a$  and  $\bar{k}_{abc\dots}$  in (20) must equal zero. Since  $\bar{e}_\mu^a$  and  $\bar{k}_{abc\dots}$  are nondynamical backgrounds, their Euler-Lagrange expressions need not vanish, as indicated in (19). Therefore, the six conditions resulting from (20) must be satisfied by the physical vierbein. In general, this is possible since with violation of LLI the vierbein has six additional degrees of freedom that would otherwise be gauge degrees of freedom.

In many cases cancellations can occur in (20) that reduce the Noether identity for observer LLI to a trivial identity. This is the case when only locally Lorentz-invariant combinations of fields under particle Lorentz transformations appear in the Lagrangian. For example if the dynamical vierbein only occurs in the combination  $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ , and the local backgrounds only appear through  $\bar{k}_{\lambda\mu\nu\dots} = \bar{e}_\lambda^a \bar{e}_\mu^b \bar{e}_\nu^c \dots \bar{k}_{abc\dots}$ , then each of these combinations is invariant under particle local Lorentz transformations, and the Noether identity reduces trivially to  $T^{\mu\nu} = T^{\nu\mu}$ . It is only when there is mixing between the dynamical and nondynamical local tensors in combinations that are not invariant under particle local Lorentz transformations that a nontrivial identity results.

As an illustrative example, if explicit breaking of LLI is caused by a term with  $\mathcal{L} = e^\mu_a \bar{e}_\mu^a$  in the action, and the matter fields are put on shell, then (20) combined with requiring  $T^{\mu\nu} = T^{\nu\mu}$  reduces to a set of symmetry conditions for the physical and background vierbeins,

$$e^\mu_a \bar{e}_{\mu b} - e^\mu_b \bar{e}_{\mu a} = 0. \quad (21)$$

These six constraints, which depend on the specific values of the background vierbein components, must be satisfied by the physical vierbein.

For example, if the physical and background vierbeins are expanded in infinitesimal linear approximations about flat backgrounds,

$$\begin{aligned} e^\mu_a &\simeq \delta^\mu_a - \frac{1}{2} h^\mu_a + \chi^\mu_a, \\ \bar{e}_{\mu a} &\simeq \eta_{\mu a} + \frac{1}{2} \bar{h}_{\mu a} + \bar{\chi}_{\mu a}, \end{aligned} \quad (22)$$

where  $h^\mu_a$  and  $\bar{h}_{\mu a}$  are symmetric components, and  $\chi^\mu_a$  and  $\bar{\chi}_{\mu a}$  are antisymmetric components, then the conditions in (21) reduce to

$$\chi_{ab} \simeq \bar{\chi}_{ab}. \quad (23)$$

Consistency in this case requires that the six antisymmetric components of the dynamical vierbein must equal the six anti-symmetric components of the background vierbein.

With explicit violation of LLI, it is possible to interpret the Noether identity stemming from observer LLI as providing coordinate-fixing conditions for the vierbein, such as (21). This is similar to how the conditions stemming from observer GCI can be interpreted as coordinate-fixing conditions for the metric. Indeed, the

conditions in (21) have been used as gauge-fixing conditions in quantum-gravity calculations where there is no Lorentz violation, but where a perturbative background-field formalism is used. In that context, the conditions (21) fix the vierbein to what is called the Deser-van Nieuwenhuizen gauge [22]. Evidently, as the above example shows, these same conditions can arise when there is explicit Lorentz breaking. However, with Lorentz violation, they are not a gauge choice. Instead, they are conditions imposed by the Noether identity.

## IV. APPLICATIONS

In this section, specific models with nondynamical background fields are examined. This permits a more detailed look at how the Noether identities impose conditions that either allow a theory to be consistent or rule it out. Models with spacetime fields and backgrounds must obey the identity in (15), while models in a vierbein formalism must obey both identities (18) and (20).

### A. Scalar and Tensor Backgrounds

To begin, consider gravitational models described using a metric formalism. In this case the Noether identity is given in (15), and as previously noted the consistency requirements that follow depend in large part on the type of background field that is present and how it couples to the metric tensor.

The most restrictive cases involve nondynamical scalar backgrounds. Because scalars do not couple directly to the metric, there are greater limitations on how the additional metric modes (which would otherwise be gauge degrees of freedom) can appear in the consistency conditions stemming from Noether's identity.

To examine this further, consider a set of nondynamical scalars,  $\bar{\phi}^A$ , labeled with an internal index  $A$ . The on-shell consistency conditions stemming from (15) then have the form

$$\frac{\delta \mathcal{L}}{\delta \bar{\phi}^A} \partial_\nu \bar{\phi}^A = 0, \quad (24)$$

where the index  $A$  is summed. It is assumed here that  $\partial_\nu \bar{\phi}^A \neq 0$  so that all four diffeomorphisms are explicitly broken by the background scalars. Evidently, when this is the case, and the derivatives are functionally independent, then the Euler-Lagrange equations,  $\frac{\delta \mathcal{L}}{\delta \bar{\phi}^A} = 0$ , must hold for the scalars despite the fact that they are nondynamical.

An example involving a single background scalar is Chern-Simons gravity in four spacetime dimensions. The Lagrangian  $\sqrt{-g} \mathcal{L} \sim \theta {}^*RR$ , where  $\theta$  is a nondynamical scalar and  ${}^*RR$  is the gravitational Pontryagin density [8, 9]. In this case, the Euler-Lagrange equation for the

scalar gives the condition

$$\frac{\delta \mathcal{L}}{\delta \theta} \sim {}^*RR = 0, \quad (25)$$

which must be satisfied by the metric. Thus, the space-time must either have a vanishing Pontryagin density or the theory is inconsistent.

In models with Lagrangian terms  $\mathcal{L}(g_{\mu\nu}, \bar{\phi}^A, \partial_\mu \bar{\phi}^A)$ , which are functions of the metric, the scalars, and first derivatives of the scalars with minimal couplings, the consistency conditions become

$$\left[ -D_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\phi}^A} + \frac{\partial \mathcal{L}}{\partial \bar{\phi}^A} \right] \partial_\nu \bar{\phi}^A = 0. \quad (26)$$

Since the scalars are nondynamical, it is the extra degrees of freedom in the metric that must satisfy these conditions. The metric in this case appears in the covariant derivatives,  $D_\mu$ , which depend on the connection,  $\Gamma_{\mu\nu}^\lambda$ .

The conditions in (26) require that four combinations of the Euler-Lagrange expressions must vanish. However, in theories with four background scalars,  $\bar{\phi}^A$ , which explicitly break all four diffeomorphisms, then the derivatives  $\partial_\nu \bar{\phi}^A$  will in general be functionally independent. In this case, the solutions for the metric that satisfy (26) must also satisfy the Euler-Lagrange equations for  $\bar{\phi}^A$  by causing the terms in brackets to vanish. This gives the appearance that the scalars  $\bar{\phi}^A$  are dynamical, since their Euler-Lagrange equations hold, but in fact they are not.

On the other hand, if potential terms  $\mathcal{L}(g_{\mu\nu}, \bar{k}_{\lambda\mu\nu\dots})$  are formed using only the metric, a background tensor  $\bar{k}_{\lambda\mu\nu\dots}$ , and no derivatives, then the consistency conditions in this case are the four equations

$$\begin{aligned} -D_\lambda \left( \frac{\partial \mathcal{L}}{\partial \bar{k}_{\lambda\mu\nu\dots}} \bar{k}_{\alpha\mu\nu\dots} \right) - D_\mu \left( \frac{\partial \mathcal{L}}{\partial \bar{k}_{\lambda\mu\nu\dots}} \bar{k}_{\lambda\alpha\nu\dots} \right) - \dots \\ + \frac{\partial \mathcal{L}}{\partial \bar{k}_{\lambda\mu\nu\dots}} D_\alpha \bar{k}_{\lambda\mu\nu\dots} = 0. \end{aligned} \quad (27)$$

In this case, with tensor backgrounds, the Euler-Lagrange expressions for the background tensors do not need to vanish, and  $\frac{\partial \mathcal{L}}{\partial \bar{k}_{\lambda\mu\nu\dots}} \neq 0$  can hold.

Notice that regardless of how many components a non-dynamical background, e.g.,  $\bar{k}_{\lambda\mu\nu\dots}$ , might have, there are always four consistency conditions that follow from the Noether identity associated with GCI. At the same time, specifying definite values for the components  $\bar{k}_{\lambda\mu\nu\dots}$  depends on the choice of a four-dimensional coordinate system.

It is possible, however, to consider a form of the background tensor that does not depend specifically on the choice of coordinate system. For example, in a different coordinate system, the components of the tensor background can be obtained from the original components by making a general coordinate transformation. The new coordinates become functions of the original coordinates, and they can therefore be written as four independent scalar functions,  $\bar{\phi}^A(x)$ , with labels  $A = 0, 1, 2, 3$ . Note

that  $\bar{\phi}^A$  are nondynamical scalars, since both the original and transformed coordinate systems are specified. The relationship between the original and transformed background tensors can then be written as

$$\bar{k}_{\lambda\mu\nu\dots}(x) = \partial_\lambda \bar{\phi}^A \partial_\mu \bar{\phi}^B \partial_\nu \bar{\phi}^C \dots \bar{k}_{ABC\dots}(\bar{\phi}). \quad (28)$$

Here,  $\bar{k}_{ABC\dots}(\bar{\phi})$  are scalar functions that have the same functional dependence as  $\bar{k}_{\lambda\mu\nu\dots}(x)$ . Notice that if  $\bar{\phi}^A = \delta_\mu^A x^\mu$ , then the new and original coordinates are the same, and  $\bar{k}_{ABC\dots}(\bar{\phi})$  equals  $\bar{k}_{\lambda\mu\nu\dots}(x)$ .

If this expression for  $\bar{k}_{\lambda\mu\nu\dots}$  is substituted into the Lagrangian, then  $\mathcal{L}$  can be considered as a function of the metric and the four nondynamical scalars,  $\bar{\phi}^A$ , with functional dependence given as  $\mathcal{L}(g_{\mu\nu}, \bar{\phi}^A, \partial_\mu \bar{\phi}^A)$ . It therefore changes from a model depending on an unspecified number of background components,  $\bar{k}_{\lambda\mu\nu\dots}$ , to one that depends only on four fixed scalars,  $\bar{\phi}^A$ , and their derivatives,  $\partial_\mu \bar{\phi}^A$ .

In making the switch from the description with  $\bar{k}_{\lambda\mu\nu\dots}$  to one depending on  $\bar{\phi}^A$ , the conditions stemming from the Noether identities change as well. Instead of the conditions in (27), where the Euler-Lagrange equations for  $\bar{k}_{\lambda\mu\nu\dots}$  need not hold, the new conditions are those in (26). In this case, the metric must provide solutions to the Euler-Lagrange equations for the scalars,  $\bar{\phi}^A$ , which gives the appearance that the scalars are then dynamical fields. However, the scalars  $\bar{\phi}^A$  remain fixed nondynamical backgrounds that are related to fixing the coordinates so that compatibility with GCI is maintained. In both (26) and (27), it is the existence of the additional metric degrees of freedom that allows these equations to hold.

## B. Stückelberg Trick

Since changing the background fields from  $\bar{k}_{\lambda\mu\nu\dots}$  to  $\bar{\phi}^A$ , gives solutions of (26) where the Euler-Lagrange equations for  $\bar{\phi}^A$  are required to hold, this makes possible a trick known as the Stückelberg trick [23]. The trick is to let the four scalars,  $\bar{\phi}^A$ , be dynamical, which restores diffeomorphism invariance. While the Stückelberg approach introduces four additional dynamical degrees of freedom, it also creates four gauge degrees of freedom by restoring diffeomorphism invariance. Thus, the number of independent degrees of freedom does not change.

With scalars  $\bar{\phi}^A$  that are now dynamical, the usual interpretation of Noether's identity associated with diffeomorphism invariance in GR becomes applicable. Notice, however, that with gauge invariance restored the individual Euler-Lagrange equations,

$$-D_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\phi}^A} + \frac{\partial \mathcal{L}}{\partial \bar{\phi}^A} = 0, \quad (29)$$

involve both the dynamical scalars and the metric. In particular, if the diffeomorphism gauge freedom is used to set the scalars equal to the coordinates, so that  $\bar{\phi}^A = \delta_\mu^A x^\mu$ , then the metric no longer has four gauge degrees



of freedom. In this case, it is again extra degrees of freedom in the metric that must give solutions of (29), which therefore parallels the theory with explicit breaking.

The Stückelberg trick is widely used in theories of massive gravity, where it appears to eliminate the awkwardness associated with having nondynamical backgrounds and explicit diffeomorphism breaking. However, it is important to realize that the reason the trick works hinges on the fact that when nondynamical scalars  $\bar{\phi}^A$  are introduced in place of  $\bar{k}_{\lambda\mu\nu\dots}$  the consistency conditions following from Noether's identity require that the Euler-Lagrange equations for the background scalars must hold. Without this requirement, it would not be possible to let the scalars be dynamical and still have an equivalent theory.

Lastly, note that even with the Stückelberg trick the original backgrounds,  $\bar{k}_{\lambda\mu\nu\dots}$ , never satisfy their Euler-Lagrange equations and do not become dynamical. The backgrounds  $\bar{k}_{\lambda\mu\nu\dots}$  become somewhat hidden in the Stückelberg approach, where they become dependent on the gauge fixing and choice of coordinates. Only the four scalar degrees of freedom become dynamical using the Stückelberg approach.

### C. Einstein-Maxwell with a Fixed Vector Current

As an example of explicit breaking that contains a dynamical matter field in addition to the metric and the nondynamical background, consider Einstein-Maxwell theory with a nondynamical external vector current. The dynamical Maxwell vector field is given as a covariant vector  $A_\mu$ , with field strength  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , while the nondynamical background is a contravariant vector  $\bar{k}^\mu$  that couples directly to  $A_\mu$ . This gives the background the form of a charge current,  $\bar{k}^\mu = J^\mu$ . However, the current  $J^\mu$  in this case is not dynamical, and it does not have backreactions.

The action in this case is

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu \bar{k}^\mu \right]. \quad (30)$$

With these definitions there is no coupling with the metric in the interaction term,  $A_\mu \bar{k}^\mu$ . Diffeomorphism invariance and LLI are explicitly broken in  $S$  due to the presence of the nondynamical background,  $\bar{k}^\mu$ . However, GCI still holds. A Noether identity with the form given in (15) therefore follows, where  $A_\mu$  replaces  $f^\psi$  and  $\bar{k}^\mu$  is the background tensor. In this context, the identity can be written as

$$\begin{aligned} g_{\nu\alpha} D_\mu (G^{\mu\nu} - T^{\mu\nu}) - D_\nu [(D_\mu F^{\mu\nu} + \bar{k}^\nu) A_\alpha] \\ + (D_\mu F^{\mu\nu} + \bar{k}^\nu) D_\alpha A_\nu \\ + D_\mu \left( \frac{\partial \mathcal{L}}{\partial \bar{k}^\alpha} \bar{k}^\mu \right) + \frac{\partial \mathcal{L}}{\partial \bar{k}^\mu} D_\alpha \bar{k}^\mu = 0. \end{aligned} \quad (31)$$

From this identity it follows that when the Einstein equations,  $G^{\mu\nu} = T^{\mu\nu}$ , and Maxwell equations,  $D_\mu F^{\mu\nu} =$

$-\bar{k}^\nu$ , both hold, the sum of the last two terms in (31) must therefore vanish. However,  $\bar{k}^\mu$  is nondynamical, and the variations with respect to it need not vanish,

$$\frac{\partial \mathcal{L}}{\partial \bar{k}^\mu} \neq 0. \quad (32)$$

Note that the background  $\bar{k}^\mu$  is different from a current  $J^\mu$  that is carried by dynamical charged particles or matter fields. With physical charge-carrying fields, the variations in (31) would be with respect to those fields, and the corresponding Euler-Lagrange expressions would vanish on shell.

Here, however, the vanishing of the last two terms in (31) becomes a constraint that the metric or vector field must satisfy in addition to their equations of motion. With  $\frac{\partial \mathcal{L}}{\partial \bar{k}^\mu} = A_\mu$ , the on-shell consistency conditions become

$$(D_\mu A_\nu) \bar{k}^\mu + A_\mu (D_\nu \bar{k}^\mu) = 0. \quad (33)$$

Writing out the covariant derivatives shows that the metric drops out, and the conditions reduce to

$$(\partial_\mu A_\nu) \bar{k}^\mu + A_\mu (\partial_\nu \bar{k}^\mu) = 0. \quad (34)$$

The vector  $A_\mu$  by itself must satisfy these conditions, and the result is therefore highly constrained and does not permit generic dynamical interactions. For example, if  $\bar{k}^\mu = \rho(t) \delta_0^\mu$ , the vector potential must obey  $\rho \partial_0 A_\nu + A_0 \partial_\nu \rho = 0$ , which no longer allows any independent dynamical degrees of freedom in  $A_\mu$ .

Note that in addition to the Noether identity associated with GCI, consistency with the Maxwell equations,  $D_\mu F^{\mu\nu} = -\bar{k}^\nu$ , also requires that  $D_\mu \bar{k}^\mu = 0$  must hold. This puts additional constraints on the theory, since the current conservation is not the result of dynamical matter equations of motion. Instead, the vanishing covariant divergence requires that  $\partial_\mu (\sqrt{-g} \bar{k}^\mu) = 0$  must hold, which either restricts the spacetime geometry or requires that  $\bar{k}^\mu$  must vanish. It is for this reason that the SME does not include terms of the form  $\bar{k}^\mu A_\mu$  in the gravity-photon sector even with spontaneous symmetry breaking [4].

As this example illustrates, adding dynamical matter fields that can interact with the nondynamical background does not necessarily ease the consistency conditions imposed by Noether's identity. In fact, it can have the opposite effect of requiring additional constraints that need to be satisfied.

### D. Massive Gravity

Interest in theories of massive gravity has increased substantially in recent years, since it was demonstrated that the models of de Rham, Gabadadze, and Tolley (dRGT) [14, 24, 25] do not contain a Boulware-Deser ghost [26]. The dRGT models can therefore be used to describe physical massive gravitons in the context of effective field theory.

The mass terms in models of massive gravity are constructed using potentials,  $\mathcal{L}(g_{\mu\nu}, \bar{k}_{\mu\nu})$ , which couple the metric with a symmetric two-tensor background, obeying  $\bar{k}_{\mu\nu} = \bar{k}_{\nu\mu}$ . The nondynamical background is needed to generate mass terms for  $g_{\mu\nu}$ , since quadratic and higher-order products cannot be formed for the metric by itself. In the original versions, a Minkowski background,  $\bar{k}_{\mu\nu} = \eta_{\mu\nu}$  was used, but it has also been shown that ghost-free models with a more general background  $\bar{k}_{\mu\nu}$  exist as well.

There are several formulations of massive gravity models, including both metric and vierbein formalisms. It is common to use a Stückelberg approach in theories of massive gravity. However, these are equivalent to the theories with nondynamical backgrounds and will not be considered separately here.

In a metric formalism, the dRGT models are defined in terms of square roots of the inverse metric contracted with the background, which can be written as

$$\sqrt{g^{\mu\alpha}\bar{k}_{\alpha\nu}} = \left( \sqrt{g^{-1}\bar{k}} \right)_{\nu}^{\mu}. \quad (35)$$

The generic form of the Noether identity associated with GCI then has the form

$$g_{\nu\alpha}D_{\mu}(G^{\mu\nu} - T^{\mu\nu}) - 2D_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\bar{k}_{\mu\nu}}\bar{k}_{\alpha\nu}\right) + \frac{\partial\mathcal{L}}{\partial\bar{k}_{\mu\nu}}D_{\alpha}\bar{k}_{\mu\nu} = 0. \quad (36)$$

Consistency in this case requires that the metric must take values that set the last two terms to zero on shell. Only then can covariant energy-momentum conservation hold.

The presence of square roots in the action requires that the field variations must be handled with care [27]. The question of whether the square-root matrices exist is relevant as well. However, in many calculations, it is either assumed that the square-root matrices exist or an ansatz form is used that permits the square root to be found. In the latter case, however, if not enough degrees of freedom are included for the metric, this can lead to inconsistency when no solutions to the Noether identity are found. For example, when a Minkowski background field is used, so that  $\bar{k}_{\mu\nu} = \eta_{\mu\nu}$ , and the physical metric  $g_{\mu\nu}$  is assumed to be spatially flat and homogeneous and isotropic, then there is no exact solution in dRGT gravity [28]. Here, this can be understood by the fact that with the assumptions being made, there are not enough degrees of freedom in the Noether identity (36) to make the last two terms vanish. However, when an alternative form for the background is used besides  $\eta_{\mu\nu}$ , which introduces additional components in the constraint equations, then an exact solution describing a spatially flat homogeneous and isotropic universe has been obtained [29].

Using a vierbein as defined in (3) introduces a natural square root for the metric. Similarly, the symmetric two-tensor background  $\bar{k}_{\mu\nu}$  can be written in terms of the

background vierbein as

$$\bar{k}_{\mu\nu} = \bar{e}_{\mu}^a \bar{e}_{\nu}^b \bar{k}_{ab}, \quad (37)$$

where  $\bar{k}_{ab} = \bar{k}_{ba}$  are the components of the background in the local frame.

The Noether identity stemming from observer LLI can then be used to look for constraints that the vierbein must satisfy, which can then shed light on the nature of the square-root matrices in dRGT models of massive gravity.

With the Lagrangian potential having the dependent form  $\mathcal{L}(e_{\mu}^a, \bar{e}_{\mu}^a, \bar{k}_{ab})$ , the resulting Noether identity is

$$(G^{\mu\nu} - T^{\mu\nu})(e_{\mu a}e_{\nu b} - e_{\mu b}e_{\nu a}) + \left(\frac{\delta\mathcal{L}}{\delta\bar{e}_{\mu}^a}\bar{e}_{\mu b} - \frac{\delta\mathcal{L}}{\delta\bar{e}_{\mu}^b}\bar{e}_{\mu a}\right) + 2\frac{\delta\mathcal{L}}{\delta\bar{k}_{cd}}(\eta_{ac}\bar{k}_{bd} - \eta_{bc}\bar{k}_{ad}) = 0. \quad (38)$$

However, for Lagrangians where the functional dependence on  $e_{\mu}^a$ ,  $\bar{e}_{\mu}^a$ , and  $\bar{k}_{ab}$  occurs only through the combinations  $\mathcal{L}(g_{\mu\nu}, \bar{k}_{\mu\nu})$ , the resulting Noether identity can be shown to reduce to a trivial identity that does not impose a constraint on the physical vierbein. This is because with  $\mathcal{L}$  depending on the combination  $\bar{k}_{\mu\nu} = \bar{e}_{\mu}^a \bar{e}_{\nu}^b \bar{k}_{ab}$ , it follows that

$$\frac{\delta\mathcal{L}}{\delta\bar{e}_{\mu}^a}\bar{e}_{\mu b} = 2\frac{\delta\mathcal{L}}{\delta\bar{k}_{cd}}\eta_{bc}\bar{k}_{ad}, \quad (39)$$

and therefore there are cancellations in (38) that reduce the identity to simply  $T^{\mu\nu} = T^{\nu\mu}$ .

In massive gravity, it is common to use a redefinition of the background vierbein that is unique for the case of a symmetric two-tensor. The implications of Noether's identity stemming from observer LLI can be investigated in this context as well.

In this approach, instead of using  $\bar{e}_{\mu}^a$  which links the components of the background in the spacetime and local frames, a different background vierbein  $\bar{v}_{\mu}^a$  is introduced [30, 31]. It is defined by

$$\bar{k}_{\mu\nu} = \bar{v}_{\mu}^a \bar{v}_{\nu}^b \eta_{ab}. \quad (40)$$

In this case, the redefined background vierbein  $\bar{v}_{\mu}^a$  gives  $\bar{k}_{\mu\nu}$  in terms of the same local Minkowski background  $\eta_{ab}$  that the metric equals in the local frame.

Notice that substituting  $\bar{v}_{\mu}^a$  and  $\eta_{ab}$ , respectively, for  $\bar{e}_{\mu}^a$  and  $\bar{k}_{ab}$  in (38) results in a simplified Noether identity. This is because the last set of terms in (38) with  $\bar{k}_{ab} = \eta_{ab}$  vanishes identically, and therefore the Noether identity reduces to

$$(G^{\mu\nu} - T^{\mu\nu})(e_{\mu a}e_{\nu b} - e_{\mu b}e_{\nu a}) + \left(\frac{\delta\mathcal{L}}{\delta\bar{v}_{\mu}^a}\bar{v}_{\mu b} - \frac{\delta\mathcal{L}}{\delta\bar{v}_{\mu}^b}\bar{v}_{\mu a}\right) = 0. \quad (41)$$

However, even in this form, as long as the Lagrangian still has the dependence  $\mathcal{L}(g_{\mu\nu}, \bar{k}_{\mu\nu})$ , with  $\bar{k}_{\mu\nu}$  now given

by (40), then a trivial Noether identity follows. In this case it is because

$$\frac{\delta \mathcal{L}}{\delta \bar{v}_\mu^a} \bar{v}_{\mu b} = \frac{\delta \mathcal{L}}{\delta \bar{v}_\mu^b} \bar{v}_{\mu a} \quad (42)$$

holds identically due to the form of the dependence.

To obtain a nontrivial Noether identity due to observer LLI in dRGT massive gravity, a Lagrangian potential that is not invariant under particle local Lorentz transformations must be used so that there is coupling between the physical and background vierbeins. An approach that is used in dRGT theories involves a matrix defined as

$$\gamma^\mu{}_\nu = e^\mu{}_a \bar{v}_\nu^a, \quad (43)$$

which is not invariant under particle local Lorentz transformations. In a vierbein formulation, the action can then be written in terms of this matrix, where the Lagrangian is formed as observer scalar combinations of  $\gamma^\mu{}_\nu$ . In four dimensions there are four independent observer scalars that can be formed as traces of products of the matrix  $\gamma^\mu{}_\nu$ . These are denoted as

$$X_n = \text{tr}[\gamma^n], \quad (44)$$

with  $n = 1, 2, 3, 4$ . The functional dependence of the Lagrangian potential can then be given as  $\mathcal{L}(X_n)$ .

With these definitions, the gravitational action can be written as

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R + \mathcal{L}(X_n) \right]. \quad (45)$$

Matter terms can be included as well. However, these do not have interactions with the nondynamical background, and for this reason they are not included here.

The Noether identity stemming from observer LLI in the action (45) then has the form

$$(G^{\mu\nu} - T^{\mu\nu})(e_{\mu a} e_{\nu b} - e_{\mu b} e_{\nu a}) + \sum_{n=1}^4 \frac{\partial \mathcal{L}}{\partial X_n} \left( \frac{\partial X_n}{\partial \bar{v}_\mu^a} \bar{v}_{\mu b} - \frac{\partial X_n}{\partial \bar{v}_\mu^b} \bar{v}_{\mu a} \right) = 0. \quad (46)$$

In this case, the identity is not trivial due to the coupling between  $e_\mu^a$  and  $\bar{v}_\mu^a$ , and constraints are imposed on the physical vierbein.

When the Einstein equations hold on shell, consistency requires that the sum in (46) must vanish. Depending on which scalars  $X_n$  are included in the action, there can be multiple solutions or branches that satisfy the consistency conditions [31]. However, a sufficient condition that results in the sum vanishing is that the physical vierbein must obey a symmetry condition,

$$e^\mu{}_a \bar{v}_{\mu b} - e^\mu{}_b \bar{v}_{\mu a} = 0. \quad (47)$$

Notice that this symmetry condition (47) has the same form as the Deser-van Nieuwenhuizen gauge-fixing condition in (21), except that here the background vierbein  $\bar{v}_\mu^a$  appears in place of  $\bar{e}_\mu^a$ .

It is also the case that when (47) holds, then the square of  $\gamma^\mu{}_\nu$  as defined in (43) obeys

$$\gamma^\mu{}_\alpha \gamma^\alpha{}_\nu = g^{\mu\alpha} \bar{k}_{\alpha\nu}. \quad (48)$$

This therefore verifies the existence of the square-root matrix for the metric,

$$\sqrt{g^{\mu\alpha} \bar{k}_{\alpha\nu}} = \gamma^\mu{}_\nu, \quad (49)$$

when the symmetry condition (47) holds.

While the metric formulation of dRGT does not have a Noether identity stemming from observer LLI, the vierbein formulation in terms of  $\gamma^\mu{}_\nu$  does, and it imposes the conditions (47) on the physical vierbein. Evidently, when these conditions are satisfied, as found originally in [30, 31], this also establishes the existence of a square-root matrix for the metric. Thus, a linkage is formed between the metric and vierbein formalisms when the symmetry condition stemming from Noether's identity is satisfied.

The effect of the observer LLI Noether identity in massive gravity defined using the symmetry-breaking matrices  $\gamma^\mu{}_\nu$  is that it imposes consistency conditions on the physical vierbein that makes the nondynamical vierbein compatible with the choice of local frame. Once compatibility is assured, then the physical and background vierbeins can be combined to give both  $g_{\mu\nu}$  and  $\bar{k}_{\mu\nu}$ . In the process, the antisymmetric components of the physical and background vierbeins drop out. However, those components are necessary in establishing that the square-root matrix in the metric formalism actually exists. In this way, the Noether identity due to observer LLI ends up playing a key role in the metric formulation.

## V. SUMMARY AND CONCLUSIONS

Gravity theories with fixed background fields break diffeomorphism invariance and LLI either explicitly or spontaneously. In the case of spontaneous breaking, the background fields arise as dynamical vacuum solutions,  $\bar{k}_{\lambda\mu\nu\dots}$ , which satisfy vacuum Euler-Lagrange equations. The interpretation of the Noether identities in theories with spontaneous spacetime symmetry breaking is therefore the same as in GR.

In contrast, when a gravitational theory contains a nondynamical background field, diffeomorphism invariance and LLI are explicitly broken, and a different form and interpretation of Noether identities emerges. First, a distinction must be made between the broken particle symmetries and the mathematical observer transformations that leave the action invariant. In particular, the observer invariances are required if a theory is to maintain observer independence. Therefore, gravity theories with a nondynamical background must still have local GCI and observer LLI as mathematical invariances in the action. These mathematical observer invariances

can then be used to obtain Noether identities that hold even when there is explicit spacetime symmetry breaking. However, an important feature in this context is that the background fields do not need to satisfy their Euler-Lagrange equations. This makes the Noether identities associated with observer independence very different from the usual case in GR.

The loss of diffeomorphism invariance and LLI also means that there are additional metric and vierbein degrees of freedom that appear in Einstein's equations in comparison to GR. These are the degrees of freedom that would normally be gauged away in GR or in theories with spontaneous spacetime symmetry breaking. It is these extra metric and vierbein modes that have to satisfy the Noether identities when there is explicit breaking. They also must absorb the backreactions that the immovable background field is unable to have, which then allows covariant energy-momentum conservation to hold.

When the nondynamical background tensor is given specified values  $\bar{k}_{\lambda\mu\nu\dots}$  with respect to a particular coordinate frame as well as specified values  $\bar{k}_{abc\dots}$  with respect to a particular local Lorentz basis, the observer invariances become fixed. The consistency of the theory then

requires that there must be compatibility between the form of the background, the choice of coordinates and local Lorentz frame, and the solutions for the metric and vierbein. It is the Noether identities associated with the local observer invariances that give these compatibility conditions.

In certain examples, the conditions imposed by Noether's identities match known gauge-fixing conditions in GR. However, this does not mean that the solutions are equivalent to solutions in GR, since the changes in the constraint structure can also result in additional degrees of freedom that are physical. Indeed, much of the interest in including background fields in gravity is aimed at finding modified theories with additional physical modes that can provide alternative approaches to solving issues related to quantum gravity, dark energy, or dark matter.

Lastly, in some cases, the Noether identities can rule out a theory completely or place restrictions on the geometry or matter dynamics [32]. In this way, they can provide a useful tool that can be used to investigate gravity theories with nondynamical backgrounds and explicit spacetime symmetry breaking.

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- [1] V.A. Kostelecký and S. Samuel, "Gravitational phenomenology in higher dimensional theories and strings," *Phys. Rev. D* **40**, 1886 (1989); "Spontaneous breaking of Lorentz symmetry in string theory," *Phys. Rev. D* **39**, 683 (1989); "Phenomenological Gravitational Constraints on Strings and Higher Dimensional Theories," *Phys. Rev. Lett.* **63**, 224 (1989).
  - [2] For reviews of experimental and theoretical approaches to violations of fundamental spacetime symmetries, see V.A. Kostelecký, ed., *CPT and Lorentz Symmetry VII* (World Scientific, Singapore, 2016) and the earlier volumes in this series; D. Mattingly and S. Liberati, "Lorentz breaking effective field theory models for matter and gravity: theory and observational constraints," arXiv:1208.1071; J. Tasson, "What do We Know about Lorentz Invariance?" *Rept. Prog. Phys.* **77** 062901 (2014); R. Bluhm, "Observable Constraints on Local Lorentz Invariance," in A. Ashtekar and V. Petkov, eds., *Springer Handbook of Spacetime* (Springer, Berlin, 2014) [arXiv:1302.1150].
  - [3] V.A. Kostelecký and R. Potting, "CPT, strings, and meson factories," *Phys. Rev. D* **51**, 3923 (1995); D. Colladay and V.A. Kostelecký, "CPT violation and the standard model," *Phys. Rev. D* **55**, 6760 (1997); "Lorentz-violating extension of the standard model," *Phys. Rev. D* **58**, 116002 (1998); V.A. Kostelecký and R. Lehnert, "Stability, causality, and Lorentz and CPT violation," *Phys. Rev. D* **63**, 065008 (2001); B. Altschul, Q.G. Bailey, and V.A. Kostelecký, "Lorentz violation with an antisymmetric tensor," *Phys. Rev. D* **81**, 065028 (2010); V.A. Kostelecký and J.D. Tasson, "Matter-gravity couplings and Lorentz violation," *Phys. Rev. D* **83**, 016013 (2011).
  - [4] V.A. Kostelecký, "Gravity, Lorentz violation, and the standard model," *Phys. Rev. D* **69**, 105009 (2004).
  - [5] R. Bluhm, "Overview of the SME: Implications and Phenomenology of Lorentz Violation," in J. Ehlers and C. Lämmerzahl, eds., *Special Relativity: Will It Survive the Next 101 Years?* (Springer, Berlin, 2006) [hep-ph/0506054].
  - [6] V.A. Kostelecký and N. Russell, "Data tables for Lorentz and CPT violation," *Rev. Mod. Phys.* **83**, 11 (2011) [arXiv:0801.0287v9].
  - [7] R. Bluhm and V.A. Kostelecký, "Spontaneous Lorentz violation, Nambu-Goldstone modes, and gravity," *Phys. Rev. D* **71**, 065008 (2005); R. Bluhm, S.-H. Fung and V.A. Kostelecký, "Spontaneous Lorentz and diffeomorphism violation, massive modes, and gravity," *Phys. Rev. D* **77**, 065020 (2008); V.A. Kostelecký and R. Potting, "Gravity from local Lorentz violation," *Gen. Rel. Grav.* **37**, 1675 (2005); "Gravity from spontaneous Lorentz violation," *Phys. Rev. D* **79**, 065018 (2009); B. Altschul, Q.G. Bailey, and V.A. Kostelecký, "Lorentz violation with an antisymmetric tensor," *Phys. Rev. D* **81**, 065028 (2010); C. Hernaski, "Spontaneous Breaking of Lorentz Symmetry with an antisymmetric tensor," arXiv:1608.00829.
  - [8] R. Bluhm, "Explicit versus spontaneous diffeomorphism breaking in gravity," *Phys. Rev. D* **91**, 065034 (2015).
  - [9] R. Jackiw and S.-Y. Pi, "Chern-Simons modification of general relativity," *Phys. Rev. D* **68**, 104012 (2003).
  - [10] For reviews discussing time variations of fundamental constants, see J.S.M. Ginges and V.V. Flambaum, "Violations of fundamental symmetries in atoms and tests of unification theories of elementary particles," *Phys. Rept.* **397**, 63 (2004); J.-P. Uzan, "Varying Constants, Gravitation and Cosmology," *Living Rev. Rel.* **14**, 2 (2011); J. Solà, "Fundamental Constants in Physics and Their

- Time Variation,” *Mod. Phys. Lett. A* **30**, 1502004 (2015).
- [11] V.A. Kostelecký, R. Lehnert, and M.J. Perry, “Spacetime-varying couplings and Lorentz violation,” *Phys. Rev. D* **68**, 123511 (2003).
  - [12] O. Bertolami, R. Lehnert, R. Potting, A. Ribeiro, “Cosmological acceleration, varying couplings, and Lorentz breaking,” *Phys. Rev. D* **69**, 083513 (2004).
  - [13] R. Bluhm, “Spacetime symmetry breaking and Einstein-Maxwell theory,” *Phys. Rev. D* **92**, 085015 (2015).
  - [14] For reviews of massive gravity, see K. Hinterbichler, “Theoretical Aspects of Massive Gravity,” *Rev. Mod. Phys.* **84**, 671 (2012); C. de Rham, “Massive Gravity,” *Living Rev. Relativity* **17**, 7 (2014).
  - [15] E. Noether, “Invariante Variationsprobleme” [Invariant Variation Problems], *Nachr. D. Knig. Gesellsch. D. Wiss.* (in German), Göttingen: Math-phys. Klasse, 1918: 235-257; English translation by M. A. Tavel, arXiv:physics/0503066.
  - [16] J.L. Anderson, *Principles of Relativity Physics* (Academic, New York, 1967).
  - [17] A. Trautman, “Conservation laws in general relativity,” in L. Witten, ed., *Gravitation: an Introduction to Current Research*, (J. Wiley, New York, 1962).
  - [18] H.R. Brown and K. Brading, “General Covariance from the Perspective of Noether’s Theorems,” *Diálogos* **79**, 59 (2002); K. Brading and H.R. Brown, “Symmetries and Noether’s theorems,” in K. Brading and E. Castelani, eds., *Symmetries in Physics* (Cambridge Univ., Cambridge, 2003).
  - [19] Y.N. Obukhov, and G.F. Rubilar, “Invariant conserved currents in gravity theories with local Lorentz and diffeomorphism symmetry,” *Phys. Rev. D* **74**, 064002 (2006); Y.N. Obukhov, F. Portales-Olivía, D. Puetfeld, and G.F. Rubilar, “Invariant conserved currents in generalized gravity,” *Phys. Rev. D* **92**, 104010 (2015).
  - [20] For passive transformations, see, e.g., S. Weinberg, *Gravitation and Cosmology* (J. Wiley, New York, 1972), pp. 361-363.
  - [21] R. Bluhm, N.L. Gagne, R. Potting, and A. Vrublevskis, “Constraints and stability in vector theories with spontaneous Lorentz violation,” *Phys. Rev. D* **77**, 125007 (2008).
  - [22] S. Deser and P. van Nieuwenhuizen, “Nonrenormalizability of the quantized Dirac-Einstein system,” *Phys. Rev. D* **10**, 411 (1974).
  - [23] N. Arkani-Hamed, H. Georgi, and M.D. Schwartz, “Effective field theory for massive gravitons and gravity in theory space,” *Annals Phys.* **305**, 96 (2003).
  - [24] C. de Rham and G. Gabadadze, “Generalization of the Fierz-Pauli Action,” *Phys. Rev. D* **82**, 044020 (2010); C. de Rham, G. Gabadadze and A.J. Tolley, “Resummation of Massive Gravity,” *Phys. Rev. Lett.* **106**, 231101 (2011); “Ghost free Massive Gravity in the Stückelberg language,” *Phys. Lett. B* **711**, 190 (2012).
  - [25] S.F. Hassan and R.A. Rosen, “Resolving the Ghost Problem in non-Linear Massive Gravity,” *Phys. Rev. Lett.* **108**, 041101 (2012); S.F. Hassan, R.A. Rosen, and A. Schmidt-May, “Ghost-free Massive Gravity with a General Reference Metric,” *JHEP* **1202**, 026 (2012).
  - [26] D.G. Boulware and S. Deser, “Can gravitation have a finite range?” *Phys. Rev. D* **6**, 3368 (1972).
  - [27] M.S. Volkov, “Exact self-accelerating cosmologies in the ghost-free massive gravity: The detailed derivation,” *Phys. Rev. D* **86**, 104022 (2012).
  - [28] G. D’Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava and A.J. Tolley, “Massive Cosmologies,” *Phys. Rev. D* **84**, 124046 (2011).
  - [29] A.E. Gumrukcuoglu, C. Lin and S. Mukohyama, “Cosmological perturbations of self-accelerating universe in nonlinear massive gravity,” *J. Cosmol. Astropart. Phys.* **03** (2012) 006; D. Langlois and A. Naruko, “Cosmological solutions of massive gravity on de Sitter,” *Class. Quant. Grav.* **29** 202001 (2012); M. Fasiello and A.J. Tolley, “Cosmological perturbations in Massive Gravity and the Huguchi bound,” *J. Cosmol. Astropart. Phys.* **11** (2012) 035; A. De Felice, A.E. Gumrukcuoglu, C. Lin, S. Mukohyama, “On the cosmology of massive gravity,” *Class. Quant. Grav.* **30** 184004 (2013).
  - [30] K. Hinterbichler and R.A. Rosen, “Interacting Spin-2 Fields,” *JHEP* **1207**, 047 (2012).
  - [31] C. Deffayet, J. Mourad, and G. Zahariade, “A note on symmetric vielbeins in bimetric, massive, perturbative and non perturbative gravities,” *JHEP* **1303**, 086 (2013); “Covariant constraints in ghost free massive gravity,” *JCAP* **1301**, 032 (2013).
  - [32] In theories where explicit spacetime symmetry breaking leads to potential inconsistencies in Riemannian geometry due to the presence of a nondynamical background, it has been suggested that using a Finsler geometry might allow a consistent description. See, for example, V.A. Kostelecký, “Riemann-Finsler geometry and Lorentz-violating kinematics,” *Phys. Lett. B* **701**, 137 (2011); V.A. Kostelecký, N. Russell, and R. Tso, “Bipartite Riemann-Finsler geometry and Lorentz violation,” *Phys. Lett. B* **716**, 470 (2012); M. Schreck, “Classical kinematics and Finsler structures for nonminimal Lorentz-violating fermions,” *Eur. Phys. J. C* **75**, 187 (2015); N. Russell, “Finsler-like structures from Lorentz-breaking classical particles,” *Phys. Rev. D* **91**, 045008 (2015).